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Scale-covariant field theories: V. The large-N limit of the self-interacting $\lambda(\varphi^2)^2$ scalar theory

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Abstract. In the large-N limit the ultraviolet singularities due to the self-interaction and the 'hard-core' change of measure are additive. For $d > 4$ dimensions the hard core is the less singular. The theory can be renormalised in the large-N limit for $d = 5$ dimensions when scale covariance is obligatory (and also in $d = 4, 3$ dimensions).

1. Introduction

In a series of innovative papers (Klauder 1979a, b, 1981, and references therein) Klauder has argued that conventionally non-renormalisable theories are potentially solvable provided we prepare the functional measures for the theory (in the path-integral formalism) appropriately.

Taking the theory of a single scalar field ($\lambda\varphi^4$ in $d > 4$ space–time dimensions, say) as an example, the path integrals that we need to evaluate are of the form (Euclidean momenta)

$$Z'[h] = \int \mathscr{D}'[\varphi] \exp -\frac{1}{\hbar}\left(A[\varphi] - \int d^d x\, h\varphi\right) \tag{1.1}$$

where A is the classical action. The measure $\mathscr{D}'[\varphi]$ differs from the translation-invariant measure $\mathscr{D}[\varphi]$ of the canonical theory in excluding certain paths accessible to the free scalar theory.

In Klauder (1981) it was argued that $\mathscr{D}'[\varphi]$ should be scale covariant. That is, that

$$\mathscr{D}'[\Lambda\varphi] = F[\Lambda]\mathscr{D}'[\varphi] \qquad \text{for } \Lambda(x) > 0, \forall x. \tag{1.2}$$

In terms of the canonical translation-invariant measure $\mathscr{D}$ we can express $\mathscr{D}'$ formally as

$$\mathscr{D}'[\varphi]_\beta = \frac{\mathscr{D}[\varphi]}{\Pi_x |\varphi(x)|^\beta} \qquad 0 < \beta \leq 1. \tag{1.3}$$

As has been shown in Klauder (1981), all $\mathscr{D}'_\beta$ give rise to the same subtracted scale-covariant branching equations that were the starting point of this analysis, and which we examined in detail in an earlier paper (Ebbutt and Rivers 1982b, to be referred to as II). Different values of the non-classical degree of freedom β correspond to quantising the scale-covariant theory in different ways.

Empirically, we can only evaluate Gaussian measures and expansions based upon them. It is no surprise that we have great difficulty in evaluating path integrals like (1.1).

The major problem is to understand how ultraviolet divergences arise in scale-covariant theories and the circumstances under which they can be controlled. In the previous two papers of this series (Ebbutt and Rivers 1982c, d, to be referred to as III and IV respectively) we have used the form (1.3) to examine some aspects of the pseudo-free scalar theory, for which

$$A_0[\varphi]=\int \mathrm{d}^dx[\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2]. \tag{1.4}$$

Our main conclusion was that, if the diagrams are organised in a way appropriate to mean-field or $1/N$ expansions (in the sense of first summing the most singular contributions) the 'hard-core' ultraviolet effects of the change of measure become additive† and easier to manage. In particular, to a first approximation, renormalisation is possible for $d\geqslant 4$ dimensions‡. A dimension-specific result like this is important in that it shows that the scale-covariant theory makes sense just when it is obligatory.

In this paper we shall use the large-N tactics of III and IV to examine the scale-covariant self-interacting scalar theory with action

$$A[\varphi]=\int \mathrm{d}^dx[\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{8}\lambda_0\varphi^4]. \tag{1.5}$$

Our aim is to look for equally dimension-dependent results.

This paper is organised as follows. In the next section we calculate the formal large-N behaviour of the scale-covariant O(N)-invariant generalisation of (1.5) with action

$$A[\boldsymbol{\varphi}]=\int \mathrm{d}^dx\left(\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2+\tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2+\frac{\lambda_0}{8N}(\boldsymbol{\varphi}^2)^2\right) \tag{1.6}$$

where $\boldsymbol{\varphi}$ denotes the N-dimensional vector representation of O(N). In the three sections following, we show how renormalisation can be performed in the large-N limit. In particular, we are looking for solutions to the scale-covariant theory for $d>4$ dimensions (and perhaps for $d=4$ dimensions) when the canonical theory fails. In the remaining section, before presenting our conclusions, we reconsider the generalised measures $\mathscr{D}'_\beta$ in the light of these results.

2. The large-N limit of the O(N) $\lambda(\varphi^2)^2$ theory

When examining the scale-covariant pseudo-free theory with scale-*invariant* measure (1.3) in III we argued that the physics was driven by the most ultraviolet singular diagrams. The approximation of retaining only the most singular diagrams becomes *exact* in the large-N limit of the O(N)-invariant pseudo-free theory. In consequence, renormalisation is possible in this limit for $d\geqslant 4$ dimensions, suggesting that the $1/N$ expansion may provide a tool for renormalisation of scale-covariant theories.

† *A priori*, the scale-invariant formalism puts emphasis on the multiplicative singularities associated with the operator-product expansion, which is much less easy to handle.

‡ And not possible for $d<4$ dimensions. The case $d=4$ is considered in this paper.

In this section we shall begin to calculate the large-N limit of the O(N)-invariant theory with classical action

$$A[\boldsymbol{\varphi}]=\int \mathrm{d}^d x\left(\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2+\tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2+\frac{\lambda_0}{8N}(\boldsymbol{\varphi}^2)^2\right) \tag{2.1}$$

where $\boldsymbol{\varphi}$ denotes the N-dimensional vector representation of scalar fields of O(N). The explicit N dependence in the bare coupling strength follows from the assumption that the significant region of configuration space is $\boldsymbol{\varphi}^2=\mathrm{O}(N)$, whence each term in A is individually O(N).

The path integral that we wish to evaluate is

$$Z'[\boldsymbol{h}]=\int \mathscr{D}'[\boldsymbol{\varphi}]\exp-\frac{1}{\hbar}\left(A[\boldsymbol{\varphi}]-\int \mathrm{d}^d x\,\boldsymbol{h}\cdot\boldsymbol{\varphi}\right) \tag{2.2}$$

where $\mathscr{D}'[\boldsymbol{\varphi}]$ is both O(N) invariant and scale *covariant* under those scale transformations $\boldsymbol{\varphi}(x)\to\Lambda(x)\boldsymbol{\varphi}(x)$ ($\Lambda(x)>0,\forall x$) that preserve O(N) invariance. In consequence, $\mathscr{D}'[\boldsymbol{\varphi}]$ must have the form†

$$\mathscr{D}'[\boldsymbol{\varphi}]_\beta=\prod_1^N\left[\frac{\mathscr{D}[\varphi_i]}{\Pi_x|N^{-1/2}\boldsymbol{\varphi}(x)|^\beta}\right] \tag{2.3}$$

$$=\left[\prod_1^N\mathscr{D}[\varphi_i]\right]\exp-\tfrac{1}{2}N\beta\delta(0)\int \mathrm{d}^d x\,\ln(\boldsymbol{\varphi}^2/N) \tag{2.4}$$

the O(N)-generalisation of (1.3).

In evaluating $Z'[\boldsymbol{h}]$ in the large-N limit we need to make the N-dependence explicit. To do this, we insert in Z' (in order) the functional identities

$$\text{constant}=\int \mathscr{D}[\rho]\exp\frac{N}{2\lambda_0\hbar}\int \mathrm{d}^d x(\mathrm{i}\rho-\tfrac{1}{2}\lambda_0\boldsymbol{\varphi}^2/N)^2 \tag{2.5}$$

and

$$\text{constant}=\int \mathscr{D}[\sigma][\delta(\boldsymbol{\varphi}^2-N\sigma)]=\int \mathscr{D}[\sigma]\mathscr{D}[\alpha]\exp-\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}^d x\,\alpha(\boldsymbol{\varphi}^2-N\sigma). \tag{2.6}$$

This gives

$$Z'[\boldsymbol{h}]=\int \mathscr{D}[\sigma]\mathscr{D}[\alpha]\mathscr{D}[\rho]\prod_1^N\mathscr{D}[\varphi_i]\exp-\frac{1}{\hbar}\left(\mathfrak{A}[\sigma,\alpha,\rho;\boldsymbol{\varphi}]-\int \boldsymbol{h}\cdot\boldsymbol{\varphi}\right) \tag{2.7}$$

with

$$\mathfrak{A}[\sigma,\alpha,\rho;\boldsymbol{\varphi}]=\int \mathrm{d}^d x[\tfrac{1}{2}\boldsymbol{\varphi}(-\nabla^2+m_0^2+\mathrm{i}\alpha+\mathrm{i}\rho)\boldsymbol{\varphi}-\tfrac{1}{2}N\mathrm{i}\alpha\sigma+N\rho^2/2\lambda_0+\tfrac{1}{2}\hbar N\beta\delta(0)\ln\sigma]. \tag{2.8}$$

We now perform the Gaussian $\boldsymbol{\varphi}$ integration to get

$$Z'[\boldsymbol{h}]=\int \mathscr{D}[\sigma]\mathscr{D}[\alpha]\mathscr{D}[\rho]\exp-\frac{1}{\hbar}\mathfrak{A}[\sigma,\alpha,\rho;\boldsymbol{h}] \tag{2.9}$$

† In III we used the augmented formalism that is only appropriate to $\beta=1$. In IV we saw that the generalisation to $\beta\neq1$ is straightforward. This will be reconsidered in § 6.

where

$$\mathfrak{A}[\sigma, \alpha, \rho; \boldsymbol{h}] = -\tfrac{1}{2}\int\int \boldsymbol{h}\cdot(-\nabla^2 + m_0^2 + \mathrm{i}\alpha + i\rho)^{-1}\boldsymbol{h} + \frac{N}{2}\int\left(-\mathrm{i}\alpha\sigma + \frac{\rho^2}{\lambda_0}\right)$$
$$+\tfrac{1}{2}\hbar N\left(\mathrm{Tr}\ln(-\nabla^2 + m_0^2 + \mathrm{i}\alpha + \mathrm{i}\rho) + \beta\delta(0)\int \ln\sigma\right). \tag{2.10}$$

Each term in (2.10) is $O(N)$. In the limit $N \to \infty$ ($\hbar$ fixed) we assume that the path integral $Z'[\boldsymbol{h}]$ is dominated by a single $\boldsymbol{h}$-dependent saddle point at $(\sigma, \alpha, \rho) = (\sigma[\boldsymbol{h}], \alpha[\boldsymbol{h}], \rho[\boldsymbol{h}])$ satisfying

$$\frac{\delta\mathfrak{A}}{\delta\sigma} = \frac{\delta\mathfrak{A}}{\delta\alpha} = \frac{\delta\mathfrak{A}}{\delta\rho} = 0. \tag{2.11}$$

It follows that with this dominance

$$Z'[\boldsymbol{h}] = \exp \hbar^{-1} W[\boldsymbol{h}] \approx C \exp -\hbar^{-1}\mathfrak{A}[\sigma[\boldsymbol{h}], \alpha[\boldsymbol{h}], \rho[\boldsymbol{h}]; \boldsymbol{h}] \tag{2.12}$$

where $\ln C$ is $O(1)$, whence

$$W[\boldsymbol{h}] = -\mathfrak{A}[\sigma[\boldsymbol{h}], \alpha[\boldsymbol{h}], \rho[\boldsymbol{h}]; \boldsymbol{h}](1 + O(N^{-1})). \tag{2.13}$$

The easiest quantity to calculate is the effective action $\Gamma[\boldsymbol{\varphi}]$, the generating functional for one $\boldsymbol{\varphi}$ irreducible Green function. This is the Legendre transform of $W[\boldsymbol{h}]$ with respect to $\varphi_i = \delta W/\delta h_i$. As an intermediate step in calculating the large-N limit of $\Gamma[\boldsymbol{\varphi}]$ we introduce the generalised effective action $\Gamma[\sigma, \alpha, \rho; \boldsymbol{\varphi}]$, the Legendre transform of $-\mathfrak{A}[\sigma, \alpha, \rho; \boldsymbol{h}]$ with respect to

$$\varphi_i = -\delta\mathfrak{A}/\delta h_i = (-\nabla^2 + m_0^2 + \mathrm{i}\alpha + \mathrm{i}\rho)^{-1} h_i. \tag{2.14}$$

That is, to order N

$$\Gamma[\sigma, \alpha, \rho; \boldsymbol{\varphi}] = \mathfrak{A}[\sigma, \alpha, \rho; \boldsymbol{h}[\boldsymbol{\varphi}]] + \int \boldsymbol{\varphi}\cdot\boldsymbol{h}[\boldsymbol{\varphi}]. \tag{2.15}$$

On inspection, it is straightforward to see that $\Gamma[\boldsymbol{\varphi}]$ is obtained from $\Gamma[\sigma, \alpha, \rho; \boldsymbol{\varphi}]$ by evaluating it at the extremum $(\sigma, \alpha, \rho) = (\sigma[\boldsymbol{\varphi}], \alpha[\boldsymbol{\varphi}], \rho[\boldsymbol{\varphi}])$ satisfying

$$\frac{\delta\Gamma}{\delta\sigma} = \frac{\delta\Gamma}{\delta\alpha} = \frac{\delta\Gamma}{\delta\sigma} = 0. \tag{2.16}$$

This follows from the fact that equations (2.16) are identical to equations (2.11) on making the substitution (2.14).

The end result is that to order N

$$\Gamma[\sigma, \alpha, \rho; \boldsymbol{\varphi}] = \int \mathrm{d}^d x[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}(m_0^2 + \mathrm{i}\alpha + \mathrm{i}\rho)\boldsymbol{\varphi}^2 + \tfrac{1}{2}N(-\mathrm{i}\alpha\sigma + \rho^2/\lambda_0) + \tfrac{1}{2}\hbar N\beta\delta(0)\ln\sigma]$$
$$+\tfrac{1}{2}\hbar N\,\mathrm{Tr}\ln(-\nabla^2 + m_0^2 + \mathrm{i}\alpha + \mathrm{i}\rho) \tag{2.17}$$

where σ, α, ρ are *independent* auxiliary fields. The expression (2.17) becomes a little more transparent if we change variables from (σ, α, ρ) to (σ, χ, ρ) with

$$\chi = m_0^2 + \mathrm{i}\alpha + \mathrm{i}\rho \tag{2.18}$$

whence

$$\Gamma[\sigma,\chi,\rho;\varphi]=\int d^dx[\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}\chi\varphi^2+N\rho^2/2\lambda_0-\tfrac{1}{2}N\sigma(\chi-m_0^2-i\rho)+\tfrac{1}{2}\hbar N\beta\delta(0)\ln\sigma]$$
$$+\tfrac{1}{2}\hbar N\,\mathrm{Tr}\ln(-\nabla^2+\chi). \quad (2.19)$$

Using this representation of the effective action we shall now show how and when the ultraviolet singularities due to the hard core and self-interaction can be renormalised.

3. The mass gap equations

As a first step we shall use (2.19) to derive the large-N effective potential $\mathscr{V}(\varphi)$, the energy density of the ground state of the theory† in which the scalar-field vector has expectation value φ. This is obtained from $\Gamma[\varphi]$ by setting φ constant in space–time and factoring out the space–time volume. On past experience, if we can renormalise $\mathscr{V}(\varphi)$ we can renormalise $\Gamma[\varphi]$, and hence all Green functions. Secondly, the symmetry of the vacuum is obtained directly from the global minimum of $\mathscr{V}(\varphi)$, which defines the ground state of the theory.

To construct $\mathscr{V}(\varphi)$ we use (2.19) to introduce the generalised potential

$$N^{-1}\mathscr{V}(\sigma,\chi,\rho;\varphi)=\tfrac{1}{2}\chi\varphi^2/N+\tfrac{1}{2}\rho^2/\lambda_0-\tfrac{1}{2}\sigma(\chi-m_0^2-i\rho)+\tfrac{1}{2}\hbar\beta\delta(0)\ln\sigma$$
$$+\tfrac{1}{2}\hbar\int đk\,\ln(k^2+\chi) \quad (3.1)$$

where $đk=(2\pi)^{-d}\,d^dk$ in d dimensions.

The effective potential $\mathscr{V}(\varphi)$ is obtained from $\mathscr{V}(\sigma,\chi,\rho;\varphi)$ by imposing the constraints

$$0=-\frac{2}{N}\frac{\partial\mathscr{V}}{\partial\sigma}=\chi-m_0^2-i\rho-\tfrac{1}{2}\hbar\beta\,\frac{\delta(0)}{\sigma} \quad (3.2)$$

$$0=\frac{2}{N}\frac{\partial\mathscr{V}}{\partial\chi}=\frac{\varphi^2}{N}-\sigma+\tfrac{1}{2}\hbar\int\frac{đk}{k^2+\chi} \quad (3.3)$$

$$0=\frac{2}{N}\frac{\partial\mathscr{V}}{\partial\rho}=\frac{2\rho}{\lambda_0}+i\sigma. \quad (3.4)$$

Using (3.3) and (3.4) to re-express ρ, σ in terms of χ we can rewrite (3.2) as

$$\chi=m_0^2+\tfrac{1}{2}\lambda_0\sigma+\tfrac{1}{2}\beta\hbar\delta(0)/\sigma$$
$$=m_0^2+\tfrac{1}{2}\lambda_0(\varphi^2/N+\hbar G(0,\chi))+\beta\delta(0)(G(0,\chi)+\varphi^2/\hbar N)^{-1} \quad (3.5)$$

where

$$G(x,m^2)=\int đk\,\frac{e^{-ikx}}{k^2+m^2} \quad (3.6)$$

is the free propagator for a scalar of mass m.

† This is strictly true only for the Minkowski theory (Coleman 1975). For the large-N limit above continuation from Euclidean to Minkowski momenta is trivial, leaving $\mathscr{V}$ unchanged.

To interpret (3.5) we observe that the true vacuum of the large-N theory satisfies

$$0=\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\varphi_i}(\boldsymbol{\varphi})=\frac{\partial\mathcal{V}}{\partial\varphi_i}(\sigma,\chi,\rho;\boldsymbol{\varphi})=\chi\varphi_i. \tag{3.7}$$

Let us first suppose that there is no symmetry breaking, so that $\varphi_i=0$. On evaluating the φ propagator from Γ of (2.18) we have† (in momentum space)

$$D_{ij}(p^2)=\delta_{ij}/(p^2+\chi_0) \tag{3.8}$$

where (from (3.5))

$$\chi_0=m_0^2+\tfrac{1}{2}\hbar\lambda_0 G(0,\chi_0)+\beta\delta(0)/G(0,\chi_0). \tag{3.9}$$

If, on the other hand, there is symmetry breaking (and the vacuum has $\mathrm{O}(N-1)$ symmetry) $\chi=0$ for the Goldstone mode. However, for the unbroken degrees of freedom, the common $(\text{mass})^2$ is still given by χ_0 of (3.9). This equation, expressing the mass of the massive φ fields self-consistently in terms of the bare parameters of the theory, is the 'mass gap' equation that we shall now examine.

For the translation-covariant canonical theory ($\beta=0$) equation (3.9) becomes the conventional large-N mass gap equation (see Coleman *et al* 1974, for example), which is essentially a Hartree self-consistent equation (Caianiello *et al* 1971) or mean-field equation (Bender *et al* 1977). It is represented diagrammatically in figure 1, where we see that it retains only the most singular contributions. On the other hand, for $\lambda_0=0$ and $\beta=1$ equation (3.9) becomes the self-consistent mass-renormalisation equation due to the 'hard-core' effects of the change in measure that we discussed in great detail in III. In this case the self-consistency arises because the scale covariance enforces homogeneity on the branching equations of the theory. It is represented diagrammatically in table 1 of III. For $\beta\neq 1$ individual diagrams in this table acquire multiplicative powers of β. We shall discuss this further in § 6.

What we see in (3.9) is that when there is self-interaction the two contributions beome exactly *additive* if (and we would anticipate, only if) we organise the diagrams as in the large-N limit.

To see the circumstances under which the gap equation (3.9) is renormalisable it is most convenient to regularise the Euclidean theory by imposing a momentum cut-off at $|\boldsymbol{k}|<\Lambda$, using the definitions (d dimensions)

$$\delta(0)_\Lambda=\int_{|\boldsymbol{k}|<\Lambda}\mathrm{đ}k=\mathrm{O}(\Lambda^d) \qquad G(0,\chi)_\Lambda=\int_{|\boldsymbol{k}|<\Lambda}\frac{\mathrm{đ}k}{k^2+\chi}=\mathrm{O}(\Lambda^{d-2}). \tag{3.10}$$

Details are given in the appendix.

The gap equation then takes the form

$$\chi_0=m_0^2+\hbar\lambda_0\mathrm{O}(\Lambda^{d-2})+\beta\mathrm{O}(\Lambda^2). \tag{3.11}$$

In the introduction we stressed that a characteristic of our solutions should be that the scale-covariant formalism works where it is obligatory. For the case of the $\lambda(\boldsymbol{\varphi}^2)^2$ theory that we are examining here, the scale-covariant formalism is obligatory for $d>4$ dimensions‡.

† The second functional derivatives of $\Gamma[\sigma,\chi,\rho;\boldsymbol{\varphi}]$ give the matrix of inverse propagators which, for $\varphi_i=0$, is diagonal in the φ sector.

‡ The argument for this is independent of the calculational scheme adopted (Klauder 1979a), be it $\hbar$ expansions, $1/N$ expansions, or whatever, relying on classical inequalities.

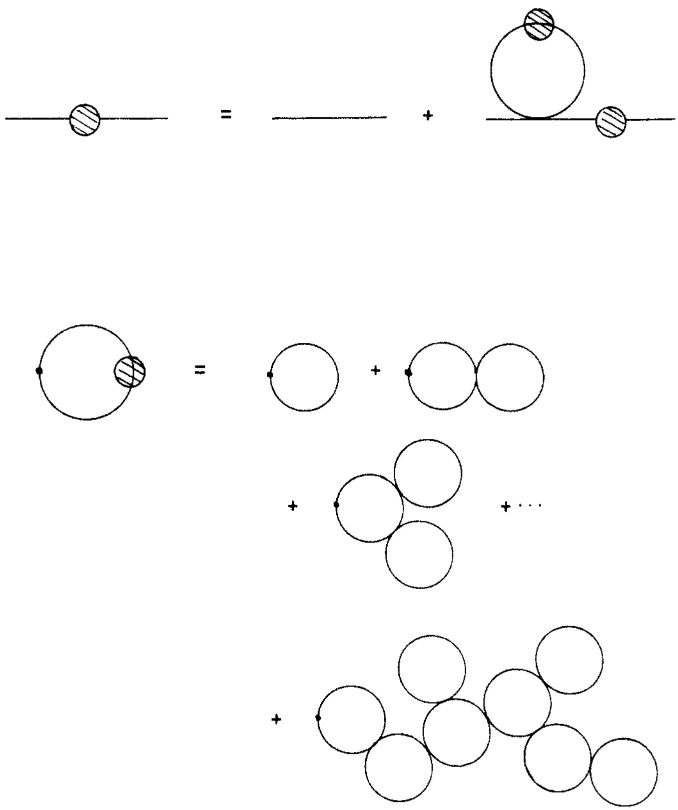

Figure 1. The Hartree-like mass gap equation for $\beta = 0$. The 'cactus' diagrams of the second line, built out of propagators $G(x, m_0^2)$, have coupling strength λ_0 at each vertex They should be contrasted to the non-polynomial diagrams of table 1 of III, to which they are added for $\beta \neq 0$.

It is just for the values $d > 4$ that, from (3.11), we see that the ultraviolet singularities due to the 'hard core' are *less singular* than the singularities due to the self-interaction. Thus, if we can control these latter singularities, there is every likelihood that the hard-core effects due to the change of measure can be subsumed in them.

We know (Rembiesa 1978) that if we ignore the change in measure† (i.e. take $\beta = 0$) the $1/N$ expansion organises the ultraviolet singularities of $\lambda(\varphi^2)^2$ in such a way that the theory can be renormalised in $d < 6$ dimensions. This suggests that, for $4 < d < 6$ dimensions, the necessarily scale-covariant theory can be renormalised.

On the other hand, for $d < 4$ dimensions the hard-core singularities are *more* singular than those due to the self-interaction. We know from IV that if $\lambda_0 = 0$, renormalisation is *not* possible and we have no expectations for $\lambda_0 \neq 0$. However, in this case (for which the scale-covariant theory is competing with an acceptable canonical theory) a positive result is less important.

To see how this is borne out in practice, we shall now renormalise the generalised φ^2-dependent mass gap equation (3.5) for *integer-d* dimensions. (We are not interested in ε expansions in this paper.)

† This has been the common practice in handling $1/N$ expansions for 'non-renormalisable' theories. Other authors besides Klauder have found this disputable (see for example Kerler 1977).

4. Renormalisation of the mass gap equation

We now examine equation (3.5) in d space–time dimensions, using the formulae of the appendix.

4.1. $d>6$ dimensions

Equation (3.5) is expressible as

$$\chi/\lambda_0 = m_0^2/\lambda_0 + \tfrac{1}{2}\varphi^2/N + \tfrac{1}{2}\hbar a_d(\Lambda^{d-2} + b_d\Lambda^{d-4}\chi + c_d\Lambda^{d-6}\chi^2 + \cdots)$$
$$+ (\tfrac{1}{2}\beta\hbar A_d/\lambda_0)(\Lambda^2 - b_d\chi + \cdots) \qquad c_d \neq 0. \tag{4.1}$$

The presence of the divergent term $\Lambda^{d-6}\chi^2$ makes it impossible to renormalise (4.1). Since this term is associated with the self-interaction it is only for the *pseudo-free* theory ($\lambda_0 = 0$) that renormalisation is possible, as was noted in IV.

4.2. $d=6$ dimensions

We now have

$$\chi/\lambda_0 = m_0^2/\lambda_0 + \tfrac{1}{2}\varphi^2/N + \tfrac{1}{2}\hbar a_6(\Lambda^4 + b_6\Lambda^2\chi + c_6\chi^2 \ln \chi/\Lambda^2 + \cdots)$$
$$+ (\tfrac{1}{2}\beta\hbar A_d/\lambda_0)(\Lambda^2 - b_6\chi + \cdots). \tag{4.2}$$

The presence of the $\chi^2 \ln \Lambda^2$ term again makes it *impossible* to renormalise (4.1) for $\lambda_0 \neq 0$. Both this and the previous result are expected.

4.3. $d=5$ dimensions

Equation (3.5) becomes

$$\frac{\chi}{\lambda_0} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\frac{\varphi^2}{N}\left(1 - \frac{2\beta\delta(0)}{\lambda_0\hbar G(0,\chi)^2}\right) + \tfrac{1}{2}\hbar G(0,\chi) + \frac{\beta\delta(0)}{\lambda_0 G(0,\chi)} \tag{4.3}$$

where we have anticipated that $\lambda_0 \to 0$ as $\Lambda \to \infty$. This gives

$$\chi\left(\frac{1}{\lambda_0}(1 - \tfrac{9}{5}\beta) + \frac{\hbar\Lambda}{24\pi^3}\right)$$
$$= \left(\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda^3}{72\pi^3} + \frac{3\beta\Lambda^2}{5\lambda_0}\right) + \frac{1}{2}\frac{\varphi^2}{N}\left(1 - \frac{216\beta\pi^3}{5\hbar\lambda_0\Lambda}\right) + \frac{\hbar}{48\pi^2}\left(1 - \frac{216\beta\pi^3}{5\hbar\lambda_0\Lambda}\right)\chi^{3/2}. \tag{4.4}$$

Defining the renormalised coupling constant λ and the renormalised mass m by

$$\frac{1}{\lambda} = \left(\frac{1}{\lambda_0}(1 - \tfrac{9}{5}\beta) + \frac{\hbar\Lambda}{24\pi^3}\right)\left(1 - \frac{216\beta\pi^3}{5\hbar\lambda_0\Lambda}\right)^{-1} \tag{4.5}$$

and

$$\frac{m^2}{\lambda} = \left(\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda^3}{72\pi^3} + \frac{3\beta\Lambda^2}{5\lambda_0}\right)\left(1 - \frac{216\beta\pi^3}{5\hbar\lambda_0\Lambda}\right)^{-1} \tag{4.6}$$

equation (4.4) becomes the finite equation

$$\chi/\lambda = m^2/\lambda + \tfrac{1}{2}\varphi^2/N + \tfrac{1}{2}\hbar G_F(0,\chi) \tag{4.7}$$

where $G_F(0, \chi)$ is the finite part of $G(0, \chi)$, given by

$$G_F(0,\chi)=G(0,\chi)-\int\frac{đk}{k^2}+\chi\int\frac{đk}{(k^2)^2}=\frac{1}{24\pi^3}\chi^{3/2}. \tag{4.8}$$

Equation (4.7), with no explicit β dependence, has just the form that we would have obtained if we had not changed the measure (Rembiesa 1978).

4.4. $d=4$ dimensions

In this case equation (3.5) is as in (4.3), but now taking the form

$$\frac{\chi}{\lambda_0}=\frac{m_0^2}{\lambda_0}+\frac{1}{2}\frac{\varphi^2}{N}\left(1-\frac{16\pi^2\beta}{\hbar\lambda_0}\right)+\frac{\hbar}{32\pi^2}(\Lambda^2+\chi\ln\chi/\Lambda^2)+\frac{\beta}{2\lambda_0}(\Lambda^2-\chi\ln\chi/\Lambda^2). \tag{4.9}$$

On introducing a subtraction point at $p^2=M^2$ (4.9) becomes

$$\chi\left[\frac{1}{\lambda_0}\left(1-\frac{\beta}{2}\ln\frac{\Lambda^2}{M^2}\right)+\frac{\hbar}{32\pi^2}\ln\frac{\Lambda^2}{M^2}\right]$$
$$=\left(\frac{m_0^2}{\lambda_0}+\frac{\hbar\Lambda^2}{32\pi^2}+\frac{\beta\Lambda^2}{2\lambda_0}\right)+\frac{1}{2}\frac{\varphi^2}{N}\left(1-\frac{16\pi^2\beta}{\hbar\lambda_0}\right)+\frac{\hbar}{32\pi^2}\left(1-\frac{16\pi^2\beta}{\hbar\lambda_0}\right)\chi\ln\frac{\chi}{M^2}. \tag{4.10}$$

Defining the renormalised coupling constant λ and renormalised mass m by

$$\frac{1}{\lambda}=\left[\frac{1}{\lambda_0}\left(1-\frac{\beta}{2}\ln\frac{\Lambda^2}{M^2}\right)+\frac{\hbar}{32\pi^2}\ln\frac{\Lambda^2}{M^2}\right]\left(1-\frac{16\pi^2\beta}{\hbar\lambda_0}\right)^{-1} \tag{4.11}$$

$$\frac{m^2}{\lambda}=\left(\frac{m_0^2}{\lambda_0}+\frac{\hbar\Lambda^2}{32\pi^2}+\frac{\beta\Lambda^2}{2\lambda_0}\right)\left(1-\frac{16\pi^2\beta}{\hbar\lambda_0}\right)^{-1} \tag{4.12}$$

gives

$$\chi/\lambda=m^2/\lambda+\tfrac{1}{2}\varphi^2/N+\tfrac{1}{2}\hbar G_F(0,\chi) \tag{4.13}$$

where $G_F(0, \chi)$ is the finite part of $G(0, \chi)$, now given by

$$G_F(0,\chi)=G(0,\chi)-\int\frac{đk}{k^2}+\chi\int\frac{đk}{k^2(k^2+M^2)}=\frac{1}{16\pi^2}\chi\ln\frac{\chi}{M^2}. \tag{4.14}$$

Again, equation (6.13) is just that for the canonical theory (Coleman *et al* 1974).

We note that if we had set $\lambda_0=0$ this would imply $\lambda=0$. This shows that there is no difficulty in renormalising the 'gap' equation for the pseudo-free theory in $d=4$ dimensions.

4.5. $d=3$ dimensions

Equation (3.5) now takes the form

$$\frac{\chi}{\lambda_0}=\frac{m_0^2}{\lambda_0}+\frac{1}{2}\frac{\varphi^2}{N}\left(1-\frac{2\beta\delta(0)}{\lambda_0\hbar G(0,\chi)^2}\right)+\frac{\beta\delta(0)}{\lambda_0\hbar^2 G(0,\chi)^3}\left(\frac{\varphi^2}{N}\right)^2$$
$$+\tfrac{1}{2}\hbar G(0,\chi)+\frac{\beta\delta(0)}{\lambda_0 G(0,\chi)}. \tag{4.15}$$

On inserting the formulae for the individual terms from the appendix we obtain

$$\frac{\chi}{\lambda_0}\left(1-\frac{\beta\pi^2}{12}\right)=\left(\frac{m_0^2}{\lambda_0}+\frac{\hbar\Lambda}{4\pi^2}+\frac{\beta\Lambda^2}{3\lambda_0}\right)+\frac{1}{2}\frac{\varphi^2}{N}\left(1-\frac{4\pi^2\beta\Lambda}{\hbar\lambda_0}\right)$$
$$-\frac{\hbar}{8\pi}\left(1-\frac{4\pi^2\beta\Lambda}{\hbar\lambda_0}\right)\chi^{1/2}+\frac{4\pi^3\beta}{3\lambda_0\hbar^2}\left(\frac{\varphi^2}{N}\right)^2. \tag{4.16}$$

If we define the renormalised coupling constant λ and the renormalised mass m by

$$\frac{1}{\lambda}=\frac{(1-\beta\pi^2/12)}{(\lambda_0-4\beta\pi^2\Lambda/\hbar)} \tag{4.17}$$

and

$$\frac{m^2}{\lambda}=\left(\frac{m_0^2}{\lambda_0}+\frac{\hbar\Lambda}{4\pi^2}+\frac{\beta\Lambda^2}{3\lambda_0}\right)\left(1-\frac{4\pi^2\beta\Lambda}{\hbar\lambda_0}\right)^{-1} \tag{4.18}$$

we see that $\lambda_0=\mathrm{O}(\Lambda)$, whence (4.16) in turn can be written as

$$\chi/\lambda=m^2/\lambda+\tfrac{1}{2}\varphi^2/N+\tfrac{1}{2}\hbar G_{\mathrm{F}}(0,\chi) \tag{4.19}$$

where $G_{\mathrm{F}}(0,\chi)$ is the finite part of $G(0,\chi)$

$$G_{\mathrm{F}}(0,\chi)=G(0,\chi)-\int\frac{\bar{d}k}{k^2}=-\frac{1}{4\pi}\chi^{1/2}. \tag{4.20}$$

We note that if we had set $\lambda_0=0$ this would imply a divergent $\lambda=\mathrm{O}(\Lambda^2)$. It is thus not possible to renormalise the pseudo-free theory in $d=3$ dimensions, as we noted in IV. The self-interaction thus seems to soften the 'hard core' so as to make it manageable.

4.6. $d=2$ dimensions

For this last case the equation (3.5) becomes

$$\frac{\chi}{\lambda_0}=\frac{m_0^2}{\lambda_0}+\frac{1}{2}\frac{\varphi^2}{N}+\frac{\hbar}{8\pi}\ln\frac{\Lambda^2}{\chi}+\frac{\beta\Lambda^2}{4\pi\lambda_0}\left(\frac{1}{4\pi}\ln\frac{\Lambda^2}{\chi}+\frac{\varphi^2}{\hbar N}\right)^{-1} \tag{4.21}$$

which it is *not* possible to renormalise.

The next step is to use the definitions in §§ 4.3–4.5 above for λ and m^2, where renormalisation is possible, to see whether it is possible to renormalise the whole of $\mathcal{V}(\varphi)$, and not just the equation (3.5) that determines its extrema. Only then can we expect to be able to renormalise $\Gamma[\varphi]$, and hence the φ Green functions which are our ultimate concern.

We conclude with a brief comment on the independent-value model (IVM) in the large-N limit, for which equation (3.5) becomes ($\beta=1$ for the IVM)

$$0=\frac{m_0^2}{\lambda_0}+\frac{1}{2}\frac{\varphi^2}{N}\left(1-\frac{2\chi^2}{\lambda_0\hbar\delta(0)}+\cdots\right)+\frac{1}{2}\frac{\hbar\delta(0)}{\chi}. \tag{4.22}$$

There is no way that we can make sense of (4.22) by adopting the multiplicative renormalisation appropriate to the IVM. It is no surprise that, in order to get a non-trivial behaviour for the O(N)-invariant IVM in the large-N limit, we are obliged

(Klauder and Narnhoffer 1976) to adopt a different explicit N dependence for λ_0 from that adopted here.

5. The renormalisation of the effective potential

The fact that we are able to renormalise the mass gap equation (3.5) which defines the extrema of the generalised effective potential (3.1) does not, in itself, guarantee that we can renormalise (3.1) on imposing (3.3)–(3.5). However, we would be surprised if we could not do so and we shall show in this section how $\mathcal{V}(\varphi)$ can indeed be renormalised in $d=3$, 4, 5 dimensions.

As a first step we eliminate ρ from $\mathcal{V}$ in terms of σ to give

$$\frac{1}{N}\mathcal{V}(\sigma,\chi;\varphi)=\tfrac{1}{2}\chi\varphi^2/N-\tfrac{1}{2}\sigma(\chi-m_0^2)+\tfrac{1}{8}\lambda_0\sigma^2+\frac{\hbar}{2}\int \mathrm{đ}k\,\ln(k^2+\chi)+\tfrac{1}{2}\beta\hbar\delta(0)\ln\sigma. \quad (5.1)$$

At this stage σ and χ are still genuinely independent auxiliary fields. On making the further substitution for σ from (3.3) we can express $\mathcal{V}(\varphi)$ as

$$\mathcal{V}(\varphi)=\mathcal{V}(\chi(\varphi);\varphi) \quad (5.2)$$

where (including all terms relevant to $d=3,4$ or 5)

$$\begin{aligned}\frac{1}{N}\mathcal{V}(\chi;\varphi)&=\tfrac{1}{2}\chi\varphi^2/N-\tfrac{1}{2}(\varphi^2/N+\hbar G(0,\chi))(\chi-m_0^2)+\tfrac{1}{8}\lambda_0(\varphi^2/N+\hbar G(0,\chi))^2\\&\quad+\frac{\hbar}{2}\int \mathrm{đ}k\,\ln(k^2+\chi)+\tfrac{1}{2}\beta\hbar\delta(0)\ln G(0,\chi)+\tfrac{1}{2}\beta\hbar\delta(0)\ln\left(1+\frac{\varphi^2}{\hbar NG(0,\chi)}\right) \qquad (5.3)\\&=\frac{\varphi^2}{2N}\left(m_0^2+\tfrac{1}{2}\hbar\lambda_0G(0,\chi)+\frac{\beta\delta(0)}{G(0,\chi)}\right)+\tfrac{1}{8}\lambda_0\left(\frac{\varphi^2}{N}\right)^2-\frac{\beta}{4\hbar}\frac{\delta(0)}{G(0,\chi)^2}\left(\frac{\varphi^2}{N}\right)^2\\&\quad+\frac{\beta}{6\hbar^2}\frac{\delta(0)}{G(0,\chi)^3}\left(\frac{\varphi^2}{N}\right)^3-\tfrac{1}{2}\hbar(\chi-m_0^2)G(0,\chi)+\tfrac{1}{8}\lambda_0\hbar^2G(0,\chi)^2\\&\quad+\tfrac{1}{2}\hbar\int \mathrm{đ}k\,\ln(k^2+\chi)+\tfrac{1}{2}\beta\hbar\delta(0)\ln G(0,\chi). \qquad (5.4)\end{aligned}$$

We now use the relations of the previous section to re-express λ_0, m_0^2 in terms of the finite quantities λ, m^2. The final renormalised $\mathcal{V}(\varphi)$ is then obtained by imposing the constraint $\chi=\chi(\varphi)$,

$$\chi(\varphi)/\lambda=m^2/\lambda+\tfrac{1}{2}\varphi^2/N+\tfrac{1}{2}\hbar G_{\mathrm{F}}(0,\chi(\varphi)) \quad (5.5)$$

common to all d.

5.1. $d=5$ dimensions

From (4.5) and (4.6) we can re-express λ_0, m_0^2 as

$$\lambda_0=-\frac{24\pi^3}{\hbar\lambda\Lambda}\left(\lambda(1-\tfrac{9}{5}\beta)+\frac{216\beta\pi^3}{5\hbar\Lambda}\right)\left(1-\frac{24\pi^3}{\hbar\lambda\Lambda}\right)^{-1}=\mathrm{O}(\Lambda^{-1}) \quad (5.6)$$

$$m_0^2=m^2(1-\tfrac{9}{5}\beta)-\tfrac{3}{5}\beta\Lambda^2-\frac{\hbar\Lambda}{72\pi^3}(\Lambda^2-3m^2)\lambda_0=\mathrm{O}(\Lambda^2). \quad (5.7)$$

It is a tedious but not difficult exercise to insert (5.6) and (5.7) in (5.4). Using the formulae of the appendix, the end result is the simple expression

$$\frac{1}{N}\mathcal{V}(\chi;\varphi)=\frac{\chi}{2}\frac{\varphi^2}{N}-\frac{\chi^2}{2\lambda}+\frac{m^2}{\lambda}\chi+\tfrac{1}{2}\hbar\left[\int \mathrm{d}k\ \ln(k^2+\chi)\right]_{\mathrm{F}} \tag{5.8}$$

where

$$\left[\int \mathrm{d}k\ \ln(k^2+\chi)\right]_{\mathrm{F}}=\int_0^{\chi}\mathrm{d}\chi'\ G_{\mathrm{F}}(0,\chi)=\frac{5}{12\pi^2}\chi^{5/2}. \tag{5.9}$$

This *β-independent* answer is the conventional translation-covariant result (Rembiesa 1978). In particular, since the constraint equation (5.5) is no more than

$$\partial\mathcal{V}(\chi;\varphi)/\partial\chi=0 \tag{5.10}$$

we can simplify the calculation by taking χ to be an *independent* auxiliary field in (5.8). The results are given in detail in Rembiesa (1978) and we refer the reader to them. What interests us is that we have been able to control the ultraviolet singularities of the hard core for a model in which it is obligatory.

5.2. d = 4 dimensions

From (4.11) and (4.12) we can re-express λ_0, m_0^2 in terms of λ, m^2. Omitting details, we find

$$\lambda_0=\frac{16\pi^2\beta}{\hbar}+\mathrm{O}((\ln\Lambda^2/M^2)^{-1}) \tag{5.11}$$

and

$$m_0^2=\mathrm{O}(\Lambda^2).$$

Repeating the exercise we again reproduce the simple *β-independent* result that

$$\mathcal{V}(\chi;\varphi)=\frac{\chi}{2}\frac{\varphi^2}{N}-\frac{\chi^2}{2\lambda}+\frac{m^2}{\lambda}\chi+\tfrac{1}{2}\hbar\left[\int \mathrm{d}k\ \ln(k^2+\chi)\right]_{\mathrm{F}} \tag{5.12}$$

where

$$\left[\int \mathrm{d}k\ \ln(k^2+\chi)\right]_{\mathrm{F}}=\int_0^{\chi}\mathrm{d}\chi'\ G_{\mathrm{F}}(0,\chi)=\frac{\chi^2}{32\pi^2}\left(\ln\frac{\chi}{M^2}-\frac{1}{2}\right). \tag{5.13}$$

As before, we can treat χ and φ as independent variables in (5.12) reproducing the orthodox results for the translation-invariant theory. We refer the reader to Coleman (1975), Kobayashi and Kugo (1975) and Abbott *et al* (1976) for details.

5.3. d = 3 dimensions

The absence of β dependence in the large-N limit of the theory in $d = 5, 4$ dimensions might seem to suggest that the effective potential is always β independent. That this is not so can be seen for this case.

As an indication of how this happens let us consider the first term in (5.4)

$$\mathscr{V}_1(\chi(\varphi);\varphi)=\frac{\varphi^2}{2N}\left(m_0^2+\frac{\hbar\lambda_0}{2}G(0,\chi)+\frac{\beta\delta(0)}{G(0,\chi)}\right) \tag{5.14}$$

$$=\frac{\varphi^2}{2N}\left[\chi-\frac{\varphi^2}{N}\left(\lambda_0-\frac{2\beta\delta(0)}{\hbar G(0,\chi)^2}\right)\right] \tag{5.15}$$

(on using (3.5)). For $d=5, 4$ dimensions

$$\lambda_0-\frac{2\beta\delta(0)_\Lambda}{\hbar G(0,\chi)_\Lambda^2}\to 0 \qquad \text{as } \Lambda\to\infty \tag{5.16}$$

to give

$$\mathscr{V}_1(\chi(\varphi);\varphi)=\tfrac{1}{2}\chi(\varphi)\varphi^2/N. \tag{5.17}$$

However, for $d=3$ dimensions we see that

$$\lambda_0-\frac{2\beta\delta(0)_\Lambda}{\hbar G(0,\chi)_\Lambda^2}\to\lambda\left(1-\frac{\beta\pi^2}{12}\right) \qquad \text{as } \Lambda\to\infty \tag{5.18}$$

giving a *β-dependent* behaviour not present in higher dimensions. A second term giving a β-dependent result is the non-zero finite term $\beta\delta(0)G(0,\chi)^{-3}(\varphi^2/N)^3$ (non-zero for $d=3$ only) in (5.4), that has no counterpart in any other term on using (5.5).

The end result of substituting for λ_0 and m_0^2 as given by (4.17) and (4.18) is to give a *β-dependent* $\mathscr{V}(\varphi)$ of the form

$$\mathscr{V}(\chi(\varphi);\varphi)=\frac{\chi}{2}\frac{\varphi^2}{N}-\frac{\chi^2}{2\lambda}+\frac{m^2}{\lambda}\chi+\tfrac{1}{2}\hbar\left[\int đk\,\ln(k^2+\chi)\right]_{\mathrm{F}}$$
$$+\frac{\pi^2\beta}{12}\left[-\frac{m^2}{\lambda}\chi-\frac{\hbar}{24\pi}\chi^{3/2}-\frac{2\pi\chi^{1/2}}{\hbar}\left(\frac{\varphi^2}{N}\right)^2+\frac{8\pi^2}{3\hbar^2}\left(\frac{\varphi^2}{N}\right)^3\right] \tag{5.19}$$

where

$$\left[\int đk\,\ln(k^2+\chi)\right]_{\mathrm{F}}=\int_0^\chi \mathrm{d}\chi'\,G_{\mathrm{F}}(0,\chi)=-\frac{1}{6\pi}\chi^{3/2}. \tag{5.20}$$

We can no longer treat χ and φ as independent in (5.19) when $\beta\neq 0$. No doubt the phase structure of the theory is β dependent. We shall not pursue this further, since we are not primarily interested in scale-covariant theories when they are not mandatory, or on the borderline of being so (i.e. $d=4$ dimensions). We are only interested in the result, presumably of greater generality, that the large-N limit (i.e. the approximation that retains the most singular ultraviolet contributions) is β dependent only when the (β-dependent) hard core is more singular than the self-interaction.

As a final comment, we were worried in IV that the $\hbar\beta\delta(0)\ln\sigma$ term might herald instability. This was shown in IV not to do so for the pseudo-free theory. What we have shown here is that, as we anticipated in IV, the self-interacting theory is equally stable in the large-N limit.

6. Measures

Let us revert to the scale-covariant theory of a single scalar field φ with action A of (1.5) and measure $\mathscr{D}'_\beta[\varphi]$ of (1.3). For $\beta\neq 0$ the generating functional $Z'[h]$ of (1.1) satisfies the subtracted equation

$$\left\{:\frac{\delta}{\delta h(x)}(-\nabla^2+m_0^2)\frac{\delta}{\delta h(x)}:+\frac{\lambda_0}{2}:\frac{\delta^4}{\delta h(x)^4}:-h(x)\frac{\delta}{\delta h(x)}\right\}Z'[h]=0 \tag{6.1}$$

where

$$:\frac{\delta^p}{\delta h(x)^p}:Z'[h]=\left(\frac{\delta^p}{\delta h(x)^p}-\frac{\delta^p Z'}{\delta h(x)^p}\bigg|_{h=0}\right)Z'[h]. \tag{6.2}$$

For $\beta=1$, when the measure is scale *invariant*, Z' satisfies the formal *unsubtracted* equation

$$\left\{\frac{\delta}{\delta h(x)}(-\nabla^2+m_0^2)\frac{\delta}{\delta h(x)}+\frac{\lambda_0}{2}\frac{\delta^4}{\delta h(x)^4}-h(x)\frac{\delta}{\delta h(x)}\right\}Z'[h]=0. \tag{6.3}$$

However, because the independent-value model (IVM) (Klauder 1975) (in which the kinetic term is dropped) still has $\beta=1$ and requires the subtraction procedure, it was argued in Klauder (1979a) that the subtracted equations should always be used and (6.3) ignored, *even* when $\beta=1$.

We saw in II that the branching equations for the unconnected Green functions

$$G(x_1x_2\ldots x_n)=\frac{\delta^n Z'[h]}{\delta h(x_1)\ldots\delta h(x_n)} \tag{6.4}$$

that follow from the subtracted equation (6.1) are *incomplete.* In order to obtain as complete a set as possible (the overall scale can never be determined) it is necessary to supplement them either with the analogue of renormalisation-group-like equations, or with additional dynamical constraints.

For $\beta=1$, this is indeed provided by the first equation of (6.3), (not obtainable from (6.1)) that

$$\lim_{x'\to x}(-\nabla_x^2+m_0^2)G_2(xx')+\tfrac{1}{2}\lambda_0 G_4(xxxx)=0. \tag{6.5}$$

It is given diagrammatically in figure 2 of paper I of this series (Ebbutt and Rivers 1982a).

Retaining $\beta=1$, we already know from III that if the scalar field has mass m

$$\lim_{x'\to x}(-\nabla^2+m_0^2)G_2(xx')=G(0,m^2)\left[m_0^2-m^2+\frac{\delta(0)}{G(0,m^2)}\right]+\text{continuum terms}. \tag{6.6}$$

Similarly, we know that

$$G_4(xxxx)=3G(0,m^2)^2+W_4(xxxx)+\text{continuum terms} \tag{6.7}$$

where W_4 is the connected four-point function.

Let us now revert to the large-N limit of the O(N) theory. Two things happen. Firstly, the scalar field loses its continuum. Secondly, $W_4(xxxx)$ is depressed by a factor of N compared with the $G(0,m^2)^2$ term, which now becomes† $NG(0,m^2)^2$. Putting in the explicit N dependence of λ_0, the large-N limit of the O(N)-invariant

† The difference in coefficients between 3 and 1 when $N=1$ is the difference between the Hartree and mean-field approximations.

generalisation of equation (6.5) becomes

$$0 = G(0, m^2)[-m^2 + m_0^2 + \delta(0)/G(0, m^2) + \tfrac{1}{2}\hbar G(0, m^2)]. \tag{6.8}$$

That is, we have rederived the gap equation (3.9).

Thus, for $\beta = 1$, the gap equation (3.9) is just the ingredient that, when added to the incomplete equations (6.1), gives the complete equations (6.3).

For $0 < \beta < 1$ the more general gap equation (3.9) still plays the role of the missing ingredient. However, we are no longer able to interpret it in terms of explicit constraints upon Green functions, such as (6.5)†. This shows the virtue of the large-N limit (and with luck, the $1/N$ expansion) in that this missing component cannot be readily identified for $\beta < 1$ by any other tactics with which we are familiar. Since β is, as yet, unspecified we have a succinct way to parametrise the degeneracy of the subtracted scale-covariant equations.

7. Conclusions

In our discussion of the large-N limit of the O(N)-invariant scale-covariant $\lambda(\varphi^2)^2$ theory several pointers to the nature of the 'hard-core' effects due to the change of measure have appeared.

Firstly, the fact that the ultraviolet singularities due to the hard core are *additive* to those of the self-interaction in the large-N limit enables us to use orthodox renormalisation techniques.

Secondly, for the $\lambda(\varphi^2)^2$ theory the hard-core ultraviolet singularities are *less* singular than those of the self-interaction just for those space–time dimensions ($d > 4$) for which the scale-covariant theory is obligatory. It is straightforward to see that this is a general result. For example, if we have a $\lambda_0 N^{1-n}(\varphi^2)^n$ theory the mass gap equation will become ($\chi_0 = m^2$ for φ mass m)

$$\chi_0 = m_0^2 + \lambda_0 G(0, \chi_0)^{n-1} + \beta\delta(0)/G(0, \chi_0). \tag{7.1}$$

In d dimensions with a $|\boldsymbol{k}| < \Lambda$ cut-off (7.1) has the form

$$\chi_0 = m_0^2 + \lambda_0 \mathrm{O}(\Lambda^{(n-1)(d-2)}) + \beta \mathrm{O}(\Lambda^2). \tag{7.2}$$

The 'hard core' is *less* singular than the self-interaction provided

$$2n < 2d/(d-2) \tag{7.3}$$

which is just the condition that the scale-covariant formalism is necessary (Klauder 1979a).

Thirdly, we have found that, provided the 'hard core' is not more singular than the self-interaction (i.e. $d \geq 4$ for $\lambda(\varphi^2)^2$), it can be absorbed in the self-interaction whenever this can be renormalised in the large-N limit. Now it happens that the critical dimension for scale covariance is essentially the critical dimension for the renormalisation of the $\hbar$ expansion (Klauder 1979a). Since the $1/N$ expansion is less singular than the $\hbar$ expansion we have a small leeway (in this case $4 < d < 6$ dimensions) in which we have a renormalisable, necessarily scale-covariant, large-N theory.

A particular consequence of the fact that the hard core can be absorbed in the self-interaction in $4 \leq d < 6$ dimensions is that the results are independent of the

† Neither is it the analogue of a renormalisation-group-like equation.

measure chosen. That is, using the measure $\mathscr{D}'_\beta$ of (1.6), we have no β dependence in the large-N limit. On the other hand, for $d<4$ dimensions, the hard-core is more singular than the self-interaction. Renormalisability is not inevitable (in particular, the pseudo-free theory is not renormalisable), but when it is (in this case for $d=3$ dimensions), the results are β dependent. We expect this result to be generalisable.

Fourthly, we have observed that, in order to get a sensible theory, the parameter β is related to the unrenormalised parameters of the theory. For example, on removing the ultraviolet cut-off in $d=4$ dimensions, we see from (5.11) that

$$\beta/\lambda_0 = \hbar/16\pi^2 \tag{7.4}$$

provided $\lambda \neq 0$. Such a 'tuning' of β is not unexpected (Klauder 1981)†.

Finally, we have observed that the β-dependent mass gap equation is just the additional piece of information needed to complement the degenerate subtracted scale-covariant equations.

We conclude with a cautionary note. The $1/N$ expansion of the canonical $\lambda(\varphi^2)^2$ theory can be pathological and insofar as the non-canonical theory presented above mimics the canonical theory, we may expect difficulties. An an example, the effective potential (5.12) for $d=4$ dimensions is potentially unstable (Linde 1976) in that, although complex for large, we have‡ (to leading order)

$$\operatorname{Re} \mathscr{V}(\varphi^2) \to -\infty \qquad \operatorname{Im} \mathscr{V}(\varphi^2)/\operatorname{Re} \mathscr{V}(\varphi^2) \to 0 \qquad \text{as } \varphi^2 \to \infty. \tag{7.5}$$

We stress that, so far, all our comments are applicable only to the large-N limit. For $d=4$ dimensions, for which β independence has only been saved by logarithms, we have no reason to expect that non-leading orders are insensitive to the change of measure.

On the other hand, for $d=5$ we have a potentially much greater chance of β independence of non-leading orders, by virtue of the $1/N$ expansion 'respecting' the change of measure, as indicated in I. This is being investigated and the results will be given elsewhere.

In the interim, it is perhaps best to consider the $1/N$ expansions as providing quantitative examples to more general qualitative expectations about non-canonical quantisation, as the (ultimately unstable) $\lambda\varphi^3$ theory proved such a useful tool in field theoretic analysis of the complex angular momentum plane.

Appendix 1. Basic formulae

A1.1. d = 2 space–time dimensions

$$G(0,\chi)_\Lambda = (4\pi)^{-1}\ln(\Lambda^2/\chi) \tag{A1}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda = \Lambda^2(\ln \Lambda^2/\chi)^{-1}. \tag{A2}$$

† This is only half the story. In principle, the first non-leading order in the $1/N$ expansion can effect an O(1) renormalisation of β, as can be seen in the pseudo-free theory (to be published elsewhere). The nature of β renormalisation for the very different interacting theory is currently under investigation.

‡ The difficulty with evaluating the arguments of orthodox instanton instability analysis is inappropriate away from the classical limit and tunnelling is not inevitable (Cant 1979) to leading order.

A1.2. d = 3 space–time dimensions

$$G(0,\chi)_\Lambda = \tfrac{1}{2}\pi^{-2}(\Lambda - \tfrac{1}{2}\pi\chi^{1/2}) \tag{A3}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda = \tfrac{1}{3}\Lambda^2(1-\pi\chi^{1/2}/2\Lambda)^{-1} = \tfrac{1}{3}(\Lambda^2 + \tfrac{1}{2}\pi\Lambda\chi^{1/2} + \tfrac{1}{4}\pi^2\chi + \mathrm{O}(\Lambda^{-1})) \tag{A4}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^2 = \tfrac{2}{3}\pi^2\Lambda(1-\pi\chi^{1/2}/2\Lambda)^{-2} = \tfrac{2}{3}\pi^2(\Lambda + \chi^{1/2} + \mathrm{O}(\Lambda^{-1})) \tag{A5}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^3 = \tfrac{4}{3}\pi^4 + \mathrm{O}(\Lambda^{-1}) \tag{A6}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^4 = \mathrm{O}(\Lambda^{-1}) \tag{A7}$$

$$\int_{|k|<\Lambda} đk\ \ln(k^2+\chi) = \tfrac{1}{2}\pi^{-1}(\Lambda\chi - \tfrac{1}{3}\pi\chi^{3/2}) + \text{constant.} \tag{A8}$$

A1.3. d = 4 space–time dimensions

$$G(0,\chi)_\Lambda = \tfrac{1}{16}\pi^{-2}(\Lambda^2 + \chi\ln(\chi/\Lambda^2) + \mathrm{O}(\Lambda^{-2})) \tag{A9}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda = \tfrac{1}{2}\Lambda^2(1+\chi\ln\chi/\Lambda^2)^{-1} = \tfrac{1}{2}(\Lambda^2 - \chi\ln\chi/\Lambda^2 + \mathrm{O}(\Lambda^{-2})) \tag{A10}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^2 = 8\pi^2 + \mathrm{O}(\Lambda^{-2}) \tag{A11}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^3 = \mathrm{O}(\Lambda^{-2}). \tag{A12}$$

A1.4. d = 5 space–time dimensions

$$G(0,\chi)_\Lambda = (\Lambda^3/36\pi^3)(1 - 3\chi/\Lambda^2 + \tfrac{3}{2}\pi\chi^{3/2}/\Lambda^3 + \mathrm{O}(\Lambda^{-4})) \tag{A13}$$

$$\begin{aligned}\delta(0)_\Lambda/G(0,\chi)_\Lambda &= \tfrac{3}{5}\Lambda^2(1 - 3\chi/\Lambda^2 + \tfrac{3}{2}\pi\chi^{3/2}/\Lambda^3 + \cdots)^{-1}\\ &= \tfrac{3}{5}(\Lambda^2 + 3\chi - \tfrac{3}{2}\pi\chi^{3/2}/\Lambda + \mathrm{O}(\Lambda^{-2}))\end{aligned} \tag{A14}$$

$$\delta(0)_\Lambda/G(0,\chi)_\Lambda^2 = 108\pi^2/5\Lambda + \mathrm{O}(\Lambda^{-2}) \tag{A15}$$

$$\int_{|k|<\Lambda} \mathrm{d}k\ \ln(k^2+\chi) = (1/36\pi^3)(\Lambda^3\chi - \tfrac{3}{2}\Lambda\chi^2 + \tfrac{3}{5}\pi\chi^{5/2}) + \text{constant.} \tag{A16}$$

A1.5. d ⩾ 6 space–time dimensions

$$\begin{aligned} d=6 \qquad & G(0,\chi)_\Lambda = a_6(\Lambda^4 + b_6\Lambda^2\chi + c_6\chi^2\ln\chi/\Lambda^2 + \cdots)\\ d>6 \qquad & G(0,\chi)_\Lambda = a_d(\Lambda^{d-2} + b_d\Lambda^{d-4}\chi + c_d\Lambda^{d-6}\chi^2 + \cdots)\end{aligned} \tag{A17}$$

$$\begin{aligned} d\geqslant 6 \qquad & \delta(0)_\Lambda/G(0,\chi)_\Lambda = A_d\Lambda^2(1 + b_d\chi/\Lambda^2 + \cdots)^{-1} = A_d(\Lambda^2 - b_d\chi + \cdots)\\ & \delta(0)_\Lambda/G(0,\chi)_\Lambda^2 = \mathrm{O}(\Lambda^{4-d}).\end{aligned} \tag{A18}$$

Unless specifically stated, logarithmic divergences are ignored in expressions $\mathrm{O}(\Lambda^n)$.

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